Complex Integration Techniques

Dr. Paul L. Bailey - December 3, 2017

We summarize the primary techniques we have obtained to compute path integrals.

Definition 1. (Path Integrals via Parameterization)
Let D be an open connected subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$. Let $\gamma : [a, b] \to D$ be a piecewise smooth path. The path integral of f along γ is defined to be

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

Result 1. (Parameterizations)
We will use the following parameterizations.

- Let $z_1, z_2 \in \mathbb{C}$. The line segment from z_1 to z_2 is parameterized as

$$\gamma : [0, 1] \to \mathbb{C} \quad \text{given by} \quad \gamma(t) = z_1 + (z_2 - z_1)t.$$

- Let $z_0 \in \mathbb{C}$ and $r > 0$. The circle of radius r about z_0 is parameterized as

$$\gamma : [0, 2\pi] \to \mathbb{C} \quad \text{given by} \quad \gamma(t) = z_0 + re^{it},$$

  where $e^{it} = \operatorname{cis} t = \cos t + i \sin t$.

Result 2. (Properties) Path integration admits the following properties.

- Path integrals are *independent of parameterization*. Recall that a *contour* is the image of a piecewise smooth path. Thus if C is a contour and α and β are paths whose image is C, we have

$$\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz.$$

  Thus we may simply write $\int_C f(z)\,dz$ for the quantity above.

- Contours are *oriented*. If $-C$ denotes the contour C except parameterized in the opposite direction, then

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

- Contours may be *concatenated* or *decomposed*. If C_1 ends where C_2 begins, and $C_1 + C_2$ denotes the contour that follows C_1 and then C_2, we have

$$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

- Path integrals are *linear*. That is, if $a, b \in \mathbb{C}$ are constants,

$$\int_C af(z) + bg(z)\,dz = a\int_C f(z)\,dz + b\int_C g(z)\,dz.$$

- Path integrals produce *arclength*. If the path γ parameterizes the contour C, the arclength of C is

$$L = \int_C |\gamma'(t)|\,dt.$$

- Path integrals are *bounded*. In fact,

$$\left|\int_C f(z)\,dz\right| \leq LM,$$

  where L is the arclength of C, and M is the maximum modulus of f along the contour C.

Result 3. (Logarithms) Let C be a positively oriented circle centered at $z_0 \in \mathbb{C}$. Then

$$\int_C \frac{1}{z - z_0}\, dz = 2\pi i.$$

Result 4. (Primitives) Let D be an open connected subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$ admit a primitive F on D. If C is a contour in D from z_1 to z_2, then

$$\int_C f(z)\, dz = F(z_2) - F(z_1).$$

In particular, if C is a closed contour in $\mathbb{C}$, then

$$\int_C f(z)\, dz = 0.$$

Moreover, the converse is also true: if $\int_C f(z)\, dz = 0$ for every closed contour C in D, then f admits a primitive in D.

Result 5. (Power functions) Let C be a positively oriented circle centered at $z_0 \in \mathbb{C}$, and let $k \in \mathbb{Z}$. Then

$$\int_C (z - z_0)^k\, dz = \begin{cases} 2\pi i & \text{if } k = -1\ ; \\ 0 & \text{if } k \neq -1\ . \end{cases}$$

Result 6. (Cauchy-Goursat) Let D be an open connected subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be analytic. Suppose that α and β are paths from z_1 to z_2 in D which are homotopic in D. Then

$$\int_\alpha f(z)\, dz = \int_\beta f(z)\, dz.$$

Result 7. (Rational Functions) Let $f(z)$ be a rational function. Let $z_1, z_2, \dots, z_n$ denote the poles of f of multiplicity 1. Then

$$f(z) = g(z) + \sum_{j=1}^{n} \frac{A_i}{z - z_j},$$

where $g(z)$ is either a polynomial or a rational function whose poles have multiplicity exceeding 1. Let C be an closed curve in $\mathbb{C}$. Then

$$\int_C f(z)\, dz = 2\pi i \sum_{j=1}^{n} A_j n(C, z_j),$$

where

$$n(C, z_j) = \frac{1}{2\pi i} \int_C \frac{1}{z - z_j}\, dz$$

is the winding number of C about z_j.

Result 8. (Cauchy's Integral Formula) Let D be an open connected set and let $f : D \to \mathbb{C}$. Let C be a simple closed curve in D such that f is analytic on and inside C. Let z_0 be inside C. Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

Result 9. (Cauchy's Integral Extension) Let D be an open connected set and let $f : D \to \mathbb{C}$. Let C be a simple closed curve in D such that f is analytic on and inside C. Let z_0 be inside C and let n be a positive integer. Then the n^{th} derivative of f exists at z_0, and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$